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# Iterative solutions to boundary-value differential equations involving reflection of the argument

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*Abstract:* This paper discusses the integral and numerical methods to a boundary value problem involving reflection of the argument. It presents an algorithm which generates an iterative sequence of approximate functions which converge to the exact solution of such problem. Convergence rate is computed and some examples are analyzed to demonstrate the effectiveness of the algorithm.

*Keywords:* Boundary-value problem, reflection of the argument, convergence rate.

## Introduction

We consider a boundary-value problem involving reflection of the argument, defined for $t \in [-1, 1]$

$$\begin{cases} X''(t) - \alpha X'(t) - g(t, X(t), X(-t)) = e(t), \\ X(-1) = X(1) = 0 \end{cases} \tag{1.1}$$

where $g:[-1, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ continuous for all $(t, x, y) \in [-1, 1] \times \mathbb{R} \times \mathbb{R}$, $g_x(t, x, y)$ and $g_y(t, x, y)$ are continuous and bounded, $e(t) \in L^1([-1, 1])$ and $\alpha$ is a non-negative real number. In [1] Gupta established the existence and uniqueness theorems to (1.1). The goal of this paper is to present Picard type of implicit methods to compute solutions to (1.1). We also present an algorithm to compute approximate solutions to (1.1).

## Iterative solutions

In this section we compute solution to the boundary value problem (1.1) in Theorem 1 and present an iterative scheme to generate a sequence of functions which converge to the exact solution to (1.1). Lemma 2 proves the convergence of the sequence of approximate functions. An algorithm is presented which helps to use numerical schemes to compute approximate solutions with desired accuracy.

**Theorem 1.** *Suppose $X(t)$ is a solution to* (1.1). *Then*

$$X(t) = -\phi(t)\int_t^1 \frac{e^{\alpha s}}{\phi(s)^2}\int_{-1}^s e^{-\alpha\tau}\phi(\tau)[g(\tau,\ X(\tau),\ X(-\tau)) + e(\tau)]\ d\tau\, ds \tag{1.2}$$

*where $\phi(t) = (e^{\alpha t} - e^{-\alpha})/\alpha$ if $\alpha > 0$ and $\phi(t) = t + 1$ if $\alpha = 0$.*

**Proof.** We will prove this theorem in two cases. In first case we assume $\alpha = 0$ and in the other case $\alpha > 0$.

*Case 1.* Let $\alpha = 0$, therefore (1.1) can be written as

$$X''(t) = g(t,\ X(t),\ X(-t)) + e(t). \tag{1.3}$$

Multiply both sides of (1.3) with $t + 1$, we get

$$(t+1)X''(t) = (t+1)(g(t,\ X(t),\ X(-t)) + e(t)). \tag{1.4}$$

Since $X(-1) = 0$, so integrating the left hand side of (1.4) from $-1$ to $t$ we have

$$\int_{-1}^t (s+1)X''(s)\ ds = (s+1)X'(s)\mid_{-1}^t - \int_{-1}^t X'(s)\ ds$$
$$= (t+1)X'(t) - X(t). \tag{1.5}$$

Therefore integrating both sides of (1.4) from $-1$ to $t$ we get

$$(t+1)X'(t) - X(t) = \int_{-1}^t (s+1)[g(s,\ X(s),\ X(-s)) + e(s)]\ ds. \tag{1.6}$$

Since

$$(t+1)X'(t) - X(t) = (t+1)^2\left[\frac{X(t)}{t+1}\right]'$$

therefore for $a > -1$

$$\int_a^t\left[\frac{X(s)}{s+1}\right]'\ ds = \frac{X(t)}{t+1} - \frac{X(a)}{a+1}. \tag{1.7}$$

From (1.6) and (1.7) we have

$$\frac{X(t)}{t+1} - \frac{X(a)}{a+1} = \int_a^t \frac{1}{(s+1)^2}\int_{-1}^s (\tau+1)[g(\tau,\ X(\tau),\ X(-\tau)) + e(\tau)]\ d\tau\, ds. \tag{1.8}$$

Since $X(1) = 0$, therefore

$$-\frac{X(a)}{a+1} = \int_a^1 \frac{1}{(s+1)^2}\int_{-1}^s (\tau+1)[g(\tau,\ X(\tau),\ X(-\tau)) + e(\tau)]\ d\tau\, ds. \tag{1.9}$$

Hence

$$\frac{X(t)}{t+1} = -\int_t^1 \frac{1}{(s+1)^2}\int_{-1}^s (\tau+1)[g(\tau,\ X(t),\ X(-\tau)) + e(\tau)]\ d\tau\, ds. \tag{1.10}$$

Which implies

$$X(t) = -(t+1)\int_t^1 \frac{1}{(s+1)^2}\int_{-1}^s (\tau+1)[g(\tau,\ X(\tau),\ X(-\tau)) + e(\tau)]\ d\tau\, ds. \tag{1.11}$$

*Case 2.* In this case we let $\alpha > 0$, therefore (1.1) can be written as

$$X''(t) - \alpha X'(t) = g(t, X(t), X(-t)) + e(t). \tag{1.12}$$

Let $\phi(t) = (e^{\alpha t} - e^{-\alpha})/\alpha$. Clearly $\phi(-1) = 0$ and $\phi''(t) - \alpha\phi'(t) = 0$. Multiply both sides of (1.12) with $\phi(t)$, we have

$$\phi(t)[X''(t) - \alpha X'(t)] = \phi(t)[g(t, X(t), X(-t)) + e(t)]. \tag{1.13}$$

Since

$$\begin{aligned}
&\int_{-1}^{t} \phi(x)(e^{-\alpha s}X'(s))' \,ds \\
&\quad = \phi(s)\, e^{-\alpha s}X'(s)|_{-1}^{t} - \int_{-1}^{t} \phi'(s)\, e^{-\alpha s}X'(s)\,ds \\
&\quad = \phi(t)\, e^{-\alpha t}X'(t) - \phi'(s)\, e^{-\alpha s}X(s)\,|_{-1}^{t} + \int_{-1}^{t} (\phi'' - \alpha\phi')\, e^{-\alpha s}X(s)\,ds \\
&\quad = \phi(t)\, e^{-\alpha t}X'(t) - \phi'(t)\, e^{-\alpha t}X(t).
\end{aligned} \tag{1.14}$$

From (1.13) and )1.14) we have

$$\phi(t)\, e^{-\alpha t}X'(t) - \phi'(t)\, e^{-\alpha t}X(t) = \int_{-1}^{t} e^{-\alpha s}\phi(s)[g(s, X(s), X(-s)) + e(s)]\,ds. \tag{1.15}$$

Which implies that

$$\phi(t)X'(t) - \phi'(t)X(t) = e^{\alpha t}\int_{-1}^{t} e^{-\alpha s}\phi(s)[g(s, X(s), X(-s)) + e(s)]\,ds. \tag{1.16}$$

Since

$$\phi(t)X'(t) - \phi'X(t) = \phi^2(t)[X(t)/\phi(t)]'$$

and for $a > -1$

$$\int_a^t \left(\frac{X(s)}{\phi(s)}\right)' ds = \frac{X(t)}{\phi(t)} - \frac{X(a)}{\phi(a)}. \tag{1.17}$$

Therefore from (1.16) and (1.17) we have

$$\frac{X(t)}{\phi(t)} - \frac{X(a)}{\phi(a)} = \int_a^t \frac{e^{\alpha s}}{\phi^2(s)} \int_{-1}^{ss} e^{-\alpha t}\phi(\tau)[g(\tau, X(\tau), X(-\tau)) + e(\tau)]\,d\tau\,ds. \tag{1.18}$$

Since $X(1) = 0$, therefore

$$X(t) = -\phi(t)\int_t^1 \frac{e^{\alpha s}}{\phi(s)^2} \int_{-1}^{s} e^{-\alpha\tau}\phi(\tau)[g(\tau, X(\tau), X(-\tau)) + e(\tau)]\,d\tau\,ds. \quad \square \tag{1.19}$$

**Lemma 1.** *Suppose* $\phi(t) = (e^{\alpha t} - e^{-\alpha})/\alpha$, *then for every* $t \in [-1, 1]$

$$\phi(t)\int_t^1 \frac{e^{\alpha s}}{\phi^2(s)} \int_{-1}^{s} e^{-\alpha t}\phi(\tau)\,d\tau\,ds \leqslant 1/M \tag{1.20}$$

*where*

$$M=\frac{\alpha(e^{2\alpha}-a)}{(t+1)(e^{2\alpha}-a)-2(e^{\alpha(t+1)}-1)}\quad \textit{with } t=\frac{1}{\alpha}\ln\left(\frac{e^{2\alpha}-1}{2\alpha}\right)-1.$$

**Proof.** Since $\phi(s)=(e^{\alpha s}-e^{-\alpha})/\alpha$, which can be written as $\phi(s)=e^{-\alpha}(e^{\alpha(s+1)}-1)/\alpha$, so the left hand side of (1.20) is

$$\begin{aligned}&\phi(t)\int_t^1\frac{e^{\alpha s}}{\phi^2(s)}\int_{-1}^s e^{-\alpha t}\phi(\tau)\,d\tau\,ds\\&=(e^{\alpha(t+1)}-1)\int_t^1\frac{e^{\alpha(s+1)}}{(e^{\alpha(s+1)}-1)^2}\int_{-1}^s(1-e^{-\alpha(\tau+1)})\,d\tau\,ds.\end{aligned}\tag{1.21}$$

But

$$\int_{-1}^s(1-e^{-\alpha(\tau-1)})\,d\tau=(s+1)+\frac{1}{\alpha}(e^{-\alpha(s+1)}-1)=\frac{1}{\alpha}\left(\alpha(s+1)+e^{-\alpha(s+1)}-1\right).\tag{1.22}$$

Therefore

$$e^{\alpha(s+1)}\int_{-1}^s(1-e^{-\alpha(\tau+1)})\,d\tau=\frac{1}{\alpha}\left[\alpha(s+1)\,e^{\alpha(s+1)}+1-e^{\alpha(s+1)}\right].\tag{1.23}$$

Which implies that

$$\begin{aligned}&\int_t^1\frac{e^{\alpha(s+1)}}{(e^{\alpha(s+1)}-1)^2}\int_{-1}^s(1-e^{-\alpha(\tau+1)})\,d\tau\,ds\\&=\frac{1}{\alpha}\int_t^1\frac{\alpha(s+1)\,e^{\alpha(s+1)}}{(e^{\alpha(s+1)}-1)^2}-\frac{1}{\alpha}\int_t^1\frac{1}{e^{\alpha(s+1)}-1}\,ds.\end{aligned}\tag{1.24}$$

Let $u=e^{\alpha(s+1)}$, then

$$\int\frac{\alpha(s+1)\,e^{\alpha(s+1)}\,ds}{(e^{\alpha(s+1)}-1)^2}=-\frac{s+1}{e^{\alpha(s+1)}-1}+\int\frac{ds}{e^{\alpha(s+1)}-1}.\tag{1.25}$$

Therefore

$$\frac{1}{\alpha}\int_t^1\left[\frac{\alpha(s+1)\,e^{\alpha(s+1)}}{(e^{\alpha(s+1)}-1)^2}-\frac{1}{e^{\alpha(s+1)}-1}\right]ds=\frac{1}{\alpha}\,\frac{(t+1)(e^{2\alpha}-1)-2(e^{\alpha(t+1)}-1)}{(e^{\alpha(t+1)}-1)(e^{2\alpha}-1)}.\tag{1.26}$$

Hence

$$\begin{aligned}&(e^{\alpha(t+1)}-1)\int_t^1\frac{e^{\alpha(s+1)}}{(e^{\alpha(s+1)}-1)^2}\int_{-1}^s(1-e^{-\alpha(\tau+1)})\,d\tau\,ds\\&=\frac{(t+1)(e^{2\alpha}-1)-2(e^{\alpha(t+1)}-1)}{\alpha(e^{2\alpha}-1)}\\&=\Phi(t).\end{aligned}\tag{1.27}$$

Since $\Phi'(t)=0$ for $t=(1/\alpha)\ln((e^{2\alpha}-1)/2\alpha)-1$ and $\Phi''(t)\leqslant 0$, implies $\Phi(t)$ is maximum for $t=1/\alpha\ln((e^{2\alpha}-1)/2\alpha)-1$. Which completes the proof. □

**Remark.** It can be shown that $M\geqslant 2$ for any $\alpha\geqslant 0$, and $M\to 2$ as $\alpha\to 0$.
Using Taylor's expansion

$$\alpha^2(e^{2\alpha}-1)=2\alpha^3+\text{higher-order terms in }\alpha,$$

$$\ln\left(\frac{e^{2\alpha}-1}{2\alpha}\right)(e^{2\alpha}-1)=\left(\frac{2\alpha}{2}+\frac{(2\alpha)^2}{3!}-\frac{(2\alpha)^2}{8}+\text{higher-order terms in }\alpha\right)$$
$$\times\left(2\alpha+\frac{(2\alpha)^2}{2!}+\text{higher-order terms in }\alpha\right)$$

and

$$e^{2\alpha}-1-2\alpha=\frac{(2\alpha)^2}{2!}+\frac{(2\alpha)^3}{3!}+\text{higher-order terms in }\alpha.$$

Therefore

$$M=\frac{\alpha(e^{2\alpha}-1)}{\frac{1}{\alpha}\ln\left(\frac{e^{2\alpha}-1}{2\alpha}\right)(e^{2\alpha}-1)-2\left(\frac{e^{2\alpha}-1-2\alpha}{2\alpha}\right)}$$
$$=\frac{2\alpha^3+\text{higher-order terms in }\alpha}{\frac{1}{8}(2\alpha)^3+\text{higher-order terms in }\alpha}=2+\text{higher-order terms in }\alpha.$$

Hence when $\alpha$ is 0 then $M$ is 2.

**Lemma 2.** *If* $\sup(|g_x|+|g_y|)<M$ (*evaluated in Lemma* 1) *and the sequence* $\{X^k(t)\}$ *is bounded by a constant* $C$, *then* $\{X^k(t)\}$ *is Cauchy, where*

$$X^{k+1}(t)=-\phi(t)\int_t^1\frac{e^{\alpha s}}{\phi(s)^2}\int_-^s e^{-\alpha\tau}\phi(\tau)\left[g(\tau,X^k(\tau),X^k(-\tau))+e(\tau)\right]\,d\tau\,ds. \tag{1.28}$$

**Proof.** Let $m<n$, subtracting $X^m$ from $X^n$ and taking absolute values on both sides, we get

$$|(X^n-X^m)(t)|\leqslant\sup(|g_x|+|g_y|)\left(\phi(t)\int_t^1\frac{e^{\alpha s}}{\phi(s)^2}\int_{-1}^s\phi(\tau)\,d\tau\,ds\right)$$
$$\times\|X^{n-1}-X^{m-1}\|_\infty \tag{1.29}$$
$$\leqslant\left(\frac{1}{M}\right)\sup(|g_x|+|g_y|)\,\|X^{n-1}-X^{m-1}\|_\infty. \tag{1.30}$$

Let $K$ denotes $\sup(|g_x|+|g_y|)/M$, therefore

$$\|X^n-X^m\|_\infty<K\|X^{n-1}-X^{m-1}\|_\infty\leqslant 2CK^m. \tag{1.31}$$

Since $0 < K < 1$, therefore the right-hand side of (1.22) can be made as small as we wish. Hence the sequence $\{X^k\}$ is Cauchy. □

**Algorithm.** Choose initial guess $X^0$

*Step 1.* For $N = 0, 1, 2, \ldots$

(a) Replace $g(t, X(t), X(-t))$ by $g(t, X^N(t), X^N(-t))$

(b) Solve $X^{N+1}(t) = T(e(t) + g(t, X^N(t), X^N(-t)))$ for $X^{N+1}(t)$, where $T$ is the integral in (1.28).

*Step 2.* Test for convergence.

*Step 3.* End iteration.

## Numerical examples

These examples establish the effectiveness of the algorithm presented in this paper. Tables indicate the convergence pattern of the iterative sequence of approximate solutions. In all these examples Trapezoidal method is used to approximate the integrals, $\epsilon_x^N$ denote $\| X - X^N \|_\infty$, $\epsilon^N$ denotes $\| X^N - X^{N-1} \|_\infty$ and denotes $\epsilon^{n-1}/\epsilon^N$ (the rate of convergence).

**Example 1.** Solve

$$\frac{d^2X}{dt^2} - \alpha\frac{dX}{dt} - \left(X(t)\,e^{-X^2(-t)} + X(-t)\,e^{-X^2(t)}\right) = e(t) \tag{1.32}$$

with boundary conditions

$$X(-1) = X(1) = 0. \tag{1.33}$$

The exact solution to (1.32), (1.33) is $X(t) = t - t^3$. In Tables 1–4 we describe the error analysis of approximate solutions and the exact solution.

From the tables we observe that when the size of $\alpha$ increases the rate of convergence of the sequence $\{X^k\}$ increases. Though the convergence to the exact solution depends on the numerical method applied to approximate integrals; here the trapezoidal method with mesh size 0.1 has been used to approximate integrals, due to which the accuracy was limited. This example does establish the effectiveness of the algorithm presented above.

Table 1

Error analysis for (1.32), (1.33) with $\alpha = 0.02$, $e(t) = 0.06t^2 - 6t - 0.02$ by using Trapezoidal method for $n = 20$, to approximate integrals

| Iteration | $\epsilon_X^N$ | $\epsilon^N$ | $R^N = \epsilon^{N-1}/\epsilon^N$ |
|---|---|---|---|
| 1 | 9.539341182 E−02 | 3.934993744 E−01 | – |
| 2 | 3.044858575 E−02 | 1.160356328 E−01 | 3.39119434 |
| 3 | 1.087370515 E−02 | 4.042559862 E−02 | 2.87035036 |
| 4 | 5.316928029 E−03 | 6.540238857 E−03 | 6.18105841 |
| 5 | 4.133045673 E−03 | 1.587755978 E−03 | 4.11917114 |

Table 2

Error analysis for (1.32), (1.33) with $\alpha = 0.2$, $e(t) = 0.6t^2 - tt^2 - 6t - 0.2$ by using the trapezoidal method for $n = 20$, to approximate integrals

| Iteration | $\epsilon_X^N$ | $\epsilon^N$ | $R^N = \epsilon^{N-1}/\epsilon^N$ |
|---|---|---|---|
| 1 | 9.487911314 E − 02 | 3.927498758 E − 01 | – |
| 2 | 2.956610918 E − 02 | 1.154061258 E − 01 | 3.40319777 |
| 3 | 1.058164239 E − 02 | 3.888699412 E − 02 | 2.96773076 |
| 4 | 5.063012242 E − 03 | 6.039381027 E − 03 | 6.43890381 |
| 5 | 4.339486361 E − 03 | 1.531071961 E − 03 | 3.94454432 |

Table 3

Error analysis for (1.32), (1.33) with $\alpha = 2$, $e(t) = 6t^2 - 6t - 2$ by using trapezoidal method for $n = 20$, to approximate integrals

| Iteration | $\epsilon_X^N$ | $\epsilon^N$ | $R^N = \epsilon^{N-1}/\epsilon^N$ |
|---|---|---|---|
| 1 | 7.729396969 E − 02 | 3.945474029 E − 01 | – |
| 2 | 1.551197469 E − 02 | 8.947459608 E − 02 | 4.40960264 |
| 3 | 8.383542299 E − 03 | 2.076348662 E − 02 | 4.30922794 |
| 4 | 7.158637047 E − 03 | 2.085953951 E − 03 | 9.95395279 |
| 5 | 7.202208042 E − 03 | 4.248511977 E − 04 | 4.90984583 |

Table 4

Error analysis for (1.32), (1.33) with $\alpha = 10$, $e(t) = 30t^2 - 6t - 10$ by using trapezoidal method for $n = 20$, to approximate integrals

| Iteration | $\epsilon_X^N$ | $\epsilon^N$ | $R^N = \epsilon^{N-1}/\epsilon^N$ |
|---|---|---|---|
| 1 | 5.811452866 E − 02 | 4.407949746 E − 01 | – |
| 2 | 4.571971297 E − 02 | 3.071986884 E − 02 | 1.434885597 E + 01 |
| 3 | 4.548209906 E − 02 | 2.296602121 E − 03 | 1.337622547 E + 01 |
| 4 | 4.550349712 E − 02 | 1.026284881 E − 04 | 2.237782288 E + 01 |
| 5 | 4.550272226 E − 02 | 2.694316208 E − 06 | 3.809073639 E + 01 |

One thing we must mention that the finite difference methods do not work effectively for problems of the type (1.1). In many cases the error dominates approximate solutions which makes the sequence $\{X^k\}$ diverge. On the other hand integral methods are far more superior to compute approximate solutions to differential equations. In [4] similar techniques have been used to solve fourth order non-linear ordinary differential equations.

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## References


[1] C.P. Gupta, Two point boundary value problems involving reflection of the argument, to appear.

[2] C.P. Gupta and J. Mawhin, Asymptotic conditions at the first two eigenvalues for periodic solutions of Lienard differential equations and inequality of E. Schmidt, *Z. Anal. Anwendungen* **3** (1984) 33–42.

[3] J. Mawhin and J.R. Ward, Periodic solutions of some forced Lienard differential equations ate resonance, *Arch. Math. (Besel)* **41** (1983) 337–351.

[4] R.K. Sharma and C.P. Gupta, Iterative solutions to non-linear fourth order differential equations through multi-integral methods, *Internat. J. Comput. Math.*, to appear.

[5] J. Wiener and A.R. Aftabizadeh, Boundary value problems for differential equations with reflection of argument, *Internat. J. Math. & Math. Sci.* **8** (1985) 151–163.